

THE TWO SIDES OF A FOURIER-STIELTJES TRANSFORM AND ALMOST IDEMPOTENT MEASURES

BY
K. DELEEuw AND Y. KATZNELSON

ABSTRACT

For measures μ on the circle T the quantities $\limsup_{n \rightarrow +\infty} |\hat{\mu}(n)|$, $\limsup_{n \rightarrow -\infty} |\hat{\mu}(n)|$ need not be equal; it is shown, however, that they are continuous with respect to each other when μ varies on bounded subsets of $M(T)$, the space of measures on T . It is also shown that measures μ which are ε -almost idempotent (i.e. $\limsup_{|n| \rightarrow \infty} |\hat{\mu}(n) - \hat{\mu}(n)^2| < \varepsilon$) are the sum of an idempotent measure and of a measure ν satisfying $\limsup_{|n| \rightarrow \infty} |\hat{\nu}(n)| < 2\varepsilon$ provided ε is small enough (as a function of $\|\mu\|$).

1. Introduction. We denote by T the circle group and by $M(T)$ the Banach space of finite complex Borel measures on T . If $\mu \in M(T)$, the Fourier-Stieltjes transform of μ is the function $\hat{\mu}$ defined on the group Z of integers by

$$\hat{\mu}(n) = \int_T e^{-int} d\mu(t), \quad n \in Z.$$

A theorem essentially* due to Rajchman states:

$$\lim_{n \rightarrow +\infty} \hat{\mu}(n) = 0 \quad \text{implies} \quad \lim_{n \rightarrow -\infty} \hat{\mu}(n) = 0.$$

If μ is discrete, $\hat{\mu}(n)$ is almost periodic on the integers and

$$\limsup_{n \rightarrow +\infty} |\hat{\mu}(n)| = \limsup_{n \rightarrow -\infty} |\hat{\mu}(n)| = \sup_n |\hat{\mu}(n)|.$$

If the support of μ is a given Helson set then, [7],

$$\limsup_{n \rightarrow +\infty} |\hat{\mu}(n)|, \quad \limsup_{n \rightarrow -\infty} |\hat{\mu}(n)|$$

are both equivalent to $\|\mu\|$.

* Rajchman's theorem ([6]) states that for $\mu \in M(T)$ $\lim_{n \rightarrow \infty} \hat{\mu}(n) = 0$, if, and only if $\lim_{n \rightarrow \infty} \hat{\nu}(n) = 0$ where $d\nu = |d\mu|$. Since $|\hat{\nu}(n)|$ is an even function of n we have: $\lim_{n \rightarrow \infty} \hat{\mu}(n) = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \hat{\nu}(n) = 0 \Leftrightarrow \lim_{n \rightarrow -\infty} \hat{\nu}(n) = 0 \Leftrightarrow \lim_{n \rightarrow -\infty} \hat{\mu}(n) = 0$.

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These facts led us to ask whether the two quantities

$$\limsup_{n \rightarrow +\infty} |\hat{\mu}(n)| \quad \limsup_{n \rightarrow -\infty} |\hat{\mu}(n)|$$

are related in general. That they need not be equal, we show by example in §4 Theorem 1, which we state below and prove in §2.2, shows, however, that they are continuous with respect to each other in bounded subsets of $M(T)$. It is a quantitative generalization of the theorem of Rajchman.

In what follows $\|\cdot\|$ is always the measure norm in $M(T)$.

THEOREM 1. *For every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ having the following property: If $\mu \in M(T)$ satisfies $\|\mu\| < 1$ and*

$$\limsup_{n \rightarrow +\infty} |\hat{\mu}(n)| < \delta,$$

then

$$\limsup_{n \rightarrow -\infty} |\hat{\mu}(n)| < \varepsilon.$$

Corollary 1, which we state and prove in §2.3 is a generalization of Theorem 1. It states that, subject to a bound on $\|\mu\|$, $\hat{\mu}(n)$ must be close to some finite set $\{\alpha_1, \dots, \alpha_m\}$ of complex numbers for $|n|$ large if $\hat{\mu}(n)$ is sufficiently close to that set for n large and positive.

The proof we give for Theorem 1 is in reality quite general. In §2.4 we indicate an abstract setting in which this argument is valid.

Helson's identification of idempotent measures on T shows the following: Let $\mu \in M(T)$ be idempotent, that is $\mu * \mu = \mu$, or equivalently,

$$\hat{\mu}(n)^2 = \hat{\mu}(n), \quad n \in \mathbf{Z}.$$

Then each of the two subsets

$$\{n: \hat{\mu}(n) = 0\} \quad \{n: \hat{\mu}(n) = 1\}$$

of \mathbf{Z} differ from periodic sets by only a finite number of elements (see [3]). Using Theorem 1, we give a generalization of this to measures that are "almost idempotent". Our result is stated as Theorem 2 and is proved in §3.1. Rather than state Theorem 2 here we give one of its consequences.

COROLLARY 2. *For any $C > 0$ there is a constant $\tau = \tau(C) > 0$ satisfying the following: Suppose that $\mu \in M(T)$ has $\|\mu\| < C$ and*

$$\limsup_{|n| \rightarrow \infty} |\hat{\mu}(n) - \hat{\mu}(n)^2| < \tau.$$

Then each of the subsets

$$\left\{ n: |\hat{\mu}(n)| < \frac{1}{10} \right\}, \quad \left\{ n: |\hat{\mu}(n) - 1| < \frac{1}{10} \right\}$$

of \mathbf{Z} differ from periodic sets by a finite number of elements.

The constant τ of Corollary 2 cannot be chosen to be independent of the bound C . We show this in §4 by giving examples of measures μ with

$$\hat{\mu}(4^m + 1) = 1, \quad m = 1, 2, 3, \dots,$$

and $\hat{\mu}(n)$ arbitrarily small for n not of the form $4^m + 1$.

2. Behavior of $\hat{\mu}$ at infinity.

2.1 Our proof of Theorem 1 depends on the following lemma.

LEMMA 1. For every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ having the following property: Let X be a set, μ a complex measure on some σ -algebra of subsets of X having measure norm $\|\mu\| \leq 1$. Let φ be a complex valued μ -measurable function on X with $|\varphi(x)| \leq 1$ (a.e. $|\mu|$). If

$$(1) \quad \left| \int_X |\varphi|^{2m} \varphi d\mu \right| < \delta, \quad m = 1, 2, \dots,$$

then

$$(2) \quad \left| \int_X \varphi d\mu \right| < \varepsilon.$$

PROOF. Fix $\varepsilon > 0$. Put $M(\varepsilon) = 4\varepsilon^{-2} \log 2/\varepsilon$ and $p_\varepsilon(x) = 1 - (1-x)^{M(\varepsilon)}$. We clearly have

$$(3) \quad 0 \leq p_\varepsilon(x) \leq 1 \quad \text{for} \quad 0 \leq x \leq 1;$$

$$(4) \quad p_\varepsilon(0) = 0;$$

$$(5) \quad \sup \left\{ |p_\varepsilon(x) - 1| : \left(\frac{\varepsilon}{2}\right)^2 \leq x \leq 1 \right\} \leq \frac{\varepsilon}{2}.$$

Because of (4), p_ε has no constant term, and thus is of the form

$$p_\varepsilon(x) = \sum_{m=1}^{M(\varepsilon)} c_{\varepsilon,m} x^m.$$

We define $\delta = \delta(\varepsilon)$ by

$$\delta(\varepsilon) = \frac{\varepsilon}{2} \left(\sum_{m=1}^{M(\varepsilon)} |c_{\varepsilon,m}| \right)^{-1} \sim \frac{\varepsilon}{2} 2^{-M(\varepsilon)}.$$

Suppose now that X , μ and φ are as in the statement of the lemma and that (1) holds. We shall prove (2).

Because of (5) and the assumption $|\varphi(x)| \leq 1$ (a.e. $|\mu|$) we have $|(p_\varepsilon(|\varphi|^2) - 1)\varphi| \leq \varepsilon/2$ and, since $\|\mu\| \leq 1$,

$$\left| \int_X \{p_\varepsilon(|\varphi|^2) - 1\} \varphi d\mu \right| \leq \frac{\varepsilon}{2},$$

that is

$$(7) \quad \left| \int_X p_\varepsilon(|\varphi|^2) \varphi d\mu - \int_X \varphi d\mu \right| \leq \frac{\varepsilon}{2}.$$

Furthermore, because of (1) and (6),

$$(8) \quad \left| \int_X p_\varepsilon(|\varphi|^2) \varphi d\mu \right| = \left| \sum_{m=1}^{M(\varepsilon)} c_{\varepsilon,m} \int_X |\varphi|^{2m} \varphi d\mu \right| \\ \leq \sum_{m=1}^{M(\varepsilon)} |c_{\varepsilon,m}| \delta(\varepsilon) < \frac{\varepsilon}{4}.$$

The inequality (2) is now a consequence of (7) and (8). This completes the proof of Lemma 1.

Note that we have proved more than Lemma 1, as we have not used the full strength of assumption (1) but only

$$\left| \int_X |\varphi|^{2m} \varphi d\mu \right| < \delta, \quad m = 1, 2, \dots, M(\varepsilon).$$

The statement of Lemma 1 will be adequate for our purposes.

2.2. We shall next prove Theorem 1. Fix ε and take $\delta = \delta(\varepsilon)$ to be that given by Lemma 1. Let μ be a measure in $M(T)$ with $\|\mu\| < 1$ and

$$(9) \quad \limsup_{n \rightarrow +\infty} |\hat{\mu}(n)| < \delta.$$

Let

$$\alpha = \limsup_{n \rightarrow -\infty} |\hat{\mu}(n)|.$$

We shall prove that $\alpha < \varepsilon$. By multiplying μ by a complex scalar of modulus 1 if necessary, we may assume that there is an increasing sequence

$$\{n_j: j = 1, 2, \dots\}$$

of positive integers so that

$$(10) \quad \alpha = \lim_{j \rightarrow \infty} \hat{\mu}(-n_j).$$

Consider the sequence

$$\{e^{in_j t} : j = 1, 2, \dots\}$$

as a subset of $L^2(|\mu|)$. This sequence is bounded in $L^2(|\mu|)$ and thus has a subsequence converging in the weak topology of $L^2(|\mu|)$ to a function φ . As a consequence, because of (10) and

$$\hat{\mu}(-n_j) = \int_T e^{in_j t} d\mu(t), \quad j = 1, 2, \dots,$$

we have

$$\int_T \varphi d\mu = \alpha.$$

Let u be any positive integer. Then φ is in the weak closure in $L^2(|\mu|)$ of the set E_u of exponentials defined by

$$E_u = \{e^{in_j t} : j \geq u\}.$$

The weak closure of a convex set in a Banach space is identical with its norm closure. (See p. 422 of [2]). Thus φ is in the norm closure in $L^2(|\mu|)$ of the set

$$\text{co } E_u = \{g : g \text{ a convex combination of functions in } E_u\}.$$

As a consequence, we can find a function g_u in $\text{co } E_u$ which satisfies

$$\int_X |g_u - \varphi|^2 d|\mu| < \frac{1}{u}.$$

We have constructed a sequence $\{g_u : u = 1, 2, \dots\}$ of trigonometric polynomials satisfying

$$(11) \quad \lim_{n \rightarrow \infty} \int_T |g_u - \varphi|^2 d|\mu| = 0$$

and

$$(12) \quad |g_u(t)| \leq 1, \quad t \in T, \quad u = 1, 2, \dots.$$

Because of (11), there is a subsequence of $\{g_u : u = 1, 2, \dots\}$ converging to φ almost everywhere with respect to $|\mu|$. Thus, because of (12), $|\varphi(t)| \leq 1$, (a.e. $|\mu|$).

We shall now apply Lemma 1. Because of Lemma 1, in order to conclude that

$$\alpha = \int_T \varphi d\mu = \left| \int_T \varphi d\mu \right| < \varepsilon,$$

we need to show that, for any positive integer m ,

$$(13) \quad \left| \int_{\mathbf{T}} \varphi^{2m+1} \bar{\varphi}^{2m} d\mu \right| < \delta.$$

Fix m . Because of (11) and (12), as $u \rightarrow \infty$,

$$g_u^{2m+1} \bar{g}_u^{2m-1}$$

converges in the norm of $L^2(|\mu|)$ to

$$\varphi^{2m+1} \bar{\varphi}^{2m-1}.$$

As a consequence, for any $\gamma > 0$, there is a positive integer N so that

$$(14) \quad \left| \int_{\mathbf{T}} g_N^{2m+1} \bar{g}_N^{2m-1} \bar{g}_v d\mu - \int_{\mathbf{T}} \varphi^{2m+1} \bar{\varphi}^{2m} d\mu \right| < \gamma \quad \text{for } v \geq N.$$

Fix N , and for any $v \geq N$ consider the trigonometric polynomial

$$(15) \quad g_N^{2m+1} \bar{g}_N^{2m-1} \bar{g}_v.$$

Since $g_v \in \text{CO } E_v$ and $g_N \in \text{CO } E_N$, the polynomial (15) is a convex combination

$$(16) \quad g_N^{2m+1} \bar{g}_N^{2m-1} \bar{g}_v = \sum_n b_{v,n} e^{-int}$$

of exponential functions. Because of (9), we can find a positive integer M so large that

$$(17) \quad |\hat{\mu}(n)| < \delta, \quad \text{all } n > M.$$

Since $g_N^{2m+1} \bar{g}_N^{2m-1}$ is a trigonometric polynomial and $g_v \in \text{CO } E_v$, it is possible to choose v so large that in (16), $b_{v,n} = 0$ unless $n > N$. For such v ,

$$(18) \quad \begin{aligned} \left| \int_{\mathbf{T}} g_N^{2m+1} \bar{g}_N^{2m-1} \bar{g}_v d\mu \right| &= \left| \sum_n b_{v,n} \int_{\mathbf{T}} e^{-int} d\mu(t) \right| \\ &= \left| \sum_n b_{v,n} \hat{\mu}(n) \right| \leq \sum_n b_{v,n} |\hat{\mu}(n)| < \delta, \end{aligned}$$

because of (17). Since, in (14), γ was arbitrary, (14) and (18) together show that (13) holds. This completes the proof of Theorem 1.

2.3. The following corollary is a generalization of Theorem 1.

COROLLARY 1. *Let $C > 0$. Suppose that $\alpha_1, \dots, \alpha_m$ are complex constants with $|\alpha_j| < C, j = 1, \dots, m$. Let $\varepsilon > 0$. Define the constant δ' by*

$$\delta' = (2C)^m \delta ((2C)^{-m} \varepsilon),$$

where δ is as in Theorem 1. If $\mu \in M(\mathbf{T})$ satisfies $\|\mu\| < C$ and

$$\limsup_{n \rightarrow +\infty} |(\hat{\mu}(n) - \alpha_1) \cdots (\hat{\mu}(n) - \alpha_m)| < \delta',$$

then

$$\limsup_{n \rightarrow -\infty} |(\hat{\mu}(n) - \alpha_1) \cdots (\hat{\mu}(n) - \alpha_m)| < \varepsilon.$$

PROOF. Apply Theorem 1 to the measure

$$(2C)^{-m}(\mu - \alpha_1 \delta_0) * \cdots * (\mu - \alpha_m \delta_0),$$

where δ_0 is the unit point mass at 0.

2.4. We next indicate a general setting in which an analogue of the proof of Theorem 1 can be given. Suppose that X is a compact Hausdorff space, A an algebra of continuous functions on X , $\|\cdot\|_A$ an algebra norm on A satisfying $\|f\|_{\text{sup}} \leq \|f\|_A$ for all $f \in A$. Assume that $f \in A$ implies $\bar{f} \in A$ and $\|\bar{f}\|_A = \|f\|_A$. Let \mathcal{A}_+ be a collection of linear subspaces of A , linearly ordered under inclusion. For $B \in \mathcal{A}_+$, denote $\{\bar{f}: f \in B\}$ by \bar{B} and define \mathcal{A}_- to be $\{\bar{B}: B \in \mathcal{A}_+\}$. The crucial property we assume for \mathcal{A}_+ is the following *swallowing property*:

For each $f \in A$ and $B_1 \in \mathcal{A}_+$, there is $B_2 \in \mathcal{A}_+$ so that $\{fg: g \in B_2\} \subset B_1$.

Let μ be a finite Borel measure on X . For $B \in \mathcal{A}_+$ or \mathcal{A}_- , define $\|\mu\|_B$ to be the norm of μ as a linear functional on B ; i.e.,

$$\|\mu\|_B = \sup \left\{ \left| \int_X f d\mu \right| : f \in B, \|f\|_A \leq 1 \right\}.$$

Define $\|\mu\|_+$ and $\|\mu\|_-$ by

$$\begin{aligned} \|\mu\|_+ &= \inf \{ \|\mu\|_B : B \in \mathcal{A}_+ \}, \\ \|\mu\|_- &= \inf \{ \|\mu_B\| : B \in \mathcal{A}_- \}. \end{aligned}$$

A proof analogous to that of Theorem 1 establishes the following:

PROPOSITION. Let $\delta = \delta(\varepsilon)$ be as in Lemma 1. If μ is a Borel measure on X with measure norm $\|\mu\| < 1$ and

$$\|\mu\|_+ < \delta,$$

then

$$\|\mu\|_- < \varepsilon.$$

For the following choices, this proposition reduces to Theorem 1. Take $X = T$, A the algebra of trigonometric polynomials on T , $\|\cdot\|_A$ defined by

$$\left\| \sum_n a_n e^{in\cdot} \right\|_A = \sum_n |a_n|.$$

For any positive integer N , let B_N be the set of trigonometric polynomials of the form

$$\sum_{n \geq N} a_n e^{int}.$$

Let $\mathcal{A}_+ = \{B_N : N = 1, 2, \dots\}$. In this case,

$$\|\mu\|_{B_N} = \sup\{|\hat{\mu}(n)| : n \geq N\},$$

so

$$\|\mu\|_+ = \limsup_{n \rightarrow +\infty} |\hat{\mu}(n)|.$$

Similarly

$$\|\mu\|_- = \limsup_{n \rightarrow -\infty} |\hat{\mu}(n)|,$$

so the proposition reduces in this case to Theorem 1.

Taking X to be a compact abelian group with ordered dual (in the sense of either [1] or [4]), analogous choices lead to a proposition relating the behavior of a Fourier Stieltjes transform at “ $+\infty$ ” with its behaviour at “ $-\infty$ ”.

Taking X to be the torus $T \times T$ and B_N to be the space of trigonometric polynomials of the form

$$\sum_{\substack{n \geq N \\ m \geq N}} a_{n,m} e^{i(nx+my)},$$

the proposition yields a relationship between

$$\limsup_{\substack{n \rightarrow +\infty \\ m \rightarrow +\infty}} |\hat{\mu}(n, m)|$$

and

$$\limsup_{\substack{n \rightarrow -\infty \\ m \rightarrow -\infty}} |\hat{\mu}(n, m)|.$$

3. Almost idempotent measures

3.1. The main result of Section 3, Theorem 2, is a quantitative generalization of Helson’s characterization of the idempotents of $M(T)$.

THEOREM 2. *For any $C > 0$ there is a $\delta = \delta(C) > 0$ satisfying the following: Suppose that $\mu \in M(T)$ has $\|\mu\| < C$ and*

$$\limsup_{|n| \rightarrow \infty} |\hat{\mu}(n) - \hat{\mu}(n)^2| < \delta.$$

Then

$$\mu = \mu_1 + \mu_2,$$

where μ_1 is idempotent with $\hat{\mu}_1$ periodic and

$$\limsup_{|n| \rightarrow \infty} |\hat{\mu}_2(n)| \leq 2 \limsup_{|n| \rightarrow \infty} |\hat{\mu}(n) - \hat{\mu}(n)^2|.$$

The proof of Theorem 2 consists of a reduction to discrete measures and continuous measures. We deal with the continuous case by means of the following lemma.

LEMMA 2. For any $C > 0$ there is a $\gamma = \gamma(C) < 1/100$ satisfying the following: Suppose that $\lambda \in M(T)$ is a continuous measure with $\|\lambda\| < C$ and, for $|n|$ sufficiently large,

$$|\hat{\lambda}(n)| < \gamma \quad \text{or} \quad \text{Re } \hat{\lambda}(n) > 1 - \gamma.$$

Then

$$\{n : |\hat{\lambda}(n)| \geq \gamma\}$$

is finite.

We shall prove Theorem 2 by assuming Lemma 2 and then later give a proof of Lemma 2 in §3.2.

Fix C . We take $\delta = \delta(C)$ to be

$$(19) \quad \delta = \frac{\gamma(C^2)}{28},$$

where γ is the function of Lemma 2. Let $\mu \in M(T)$ be a measure with $\|\mu\| < C$ and define σ by

$$(20) \quad \sigma = \limsup_{|n| \rightarrow \infty} |\hat{\mu}(n) - \hat{\mu}(n)^2|.$$

We shall show that the assumption

$$(21) \quad \sigma < \delta$$

leads to the conclusion of Theorem 2. Note that $\gamma(C^2) < 1/100$, so $\sigma < 1/1000$.

Let $\mu = \mu_c + \mu_d$ be the decomposition of μ into its continuous and discrete parts. We first show that μ_d is “nearly” idempotent in the sense that

$$(22) \quad \sup_n |\hat{\mu}_d(n) - \hat{\mu}_d(n)^2| \leq 4\sigma.$$

Because of (20), there is a constant N_0 so that $|n| > N_0$ implies $\hat{\mu}(n)$ is within 2σ of either 0 or 1. Since μ_c is a continuous measure, the mean value of the func-

tion $|\hat{\mu}_c|^2$ on \mathbf{Z} is 0. (See p. 42 of [5].) Thus, there is a set J of integers having density zero so that

$$|\hat{\mu}_c(n)| < \sigma, \quad n \notin J.$$

As a consequence, if $n \notin J$ and $|n| > N_0$, then $\hat{\mu}(n) = \hat{\mu}_d(n) - \hat{\mu}_c(n)$ is within 3σ of 0 or 1, and thus

$$(23) \quad |\hat{\mu}_d(n) - \hat{\mu}_d(n)^2| < 4\sigma, \quad |n| > N_0, \quad n \notin J.$$

Since $\hat{\mu}_d$ is an almost periodic function on \mathbf{Z} , (22) follows from (23).

We next apply (22) in order to construct the idempotent measure μ_1 . Since $4\sigma < 1/20$, \mathbf{Z} is the disjoint union of the two subsets

$$\mathbf{Z}_0 = \{n: |\hat{\mu}_d(n)| < 1/10\}$$

and

$$\mathbf{Z}_1 = \{n: |\hat{\mu}_d(n) - 1| < 1/10\}.$$

Because $\hat{\mu}_d$ is almost periodic on \mathbf{Z} , any $1/10$ -almost period for $\hat{\mu}_d$ must be a *period* for each of the sets \mathbf{Z}_0 and \mathbf{Z}_1 . Take μ_1 to be the idempotent measure in $M(\mathbf{T})$ defined by

$$(24) \quad \hat{\mu}_1(n) = \begin{cases} 0, & n \in \mathbf{Z}_0 \\ 1, & n \in \mathbf{Z}_1 \end{cases}$$

Because of (22), for any integer m , $\hat{\mu}_d(m)$ is within 5σ of either 0 or 1. In the first case, $\hat{\mu}_1(m) = 0$ and in the second, $\hat{\mu}_1(m) = 1$. This proves

$$(25) \quad \sup_n |\hat{\mu}_d(n) - \hat{\mu}_1(n)| < 5\sigma.$$

If μ_2 is defined by $\mu_2 = \mu - \mu_1$, to complete the proof of Theorem 2 it remains only to show that

$$\limsup_{|n| \rightarrow \infty} |\hat{\mu}_2(n)| < 2\sigma.$$

Now, by (20), (24) and the definition of μ_2 it is clear that $\hat{\mu}_2(n)$ is, for large $|n|$, within 2σ of -1 , 0, or 1; so that it is enough to prove that

$$(26) \quad \limsup_{|n| \rightarrow \infty} |\hat{\mu}_2(n)| \leq 12\sigma.$$

Since $\mu_2 = \mu - \mu_1 = (\mu - \mu_d) + (\mu_d - \mu_1) = \mu_c + (\mu_d - \mu_1)$,

$$\limsup_{|n| \rightarrow \infty} |\hat{\mu}_2(n)| \leq \limsup_{|n| \rightarrow \infty} |\hat{\mu}_c(n)| + \sup_n |\hat{\mu}_d(n) - \hat{\mu}_1(n)|.$$

Hence, because of (25), the proof of (26) and thus of Theorem 2 will be complete when we have shown that

$$(27) \quad \limsup_{|n| \rightarrow \infty} |\hat{\mu}_c(n)| \leq 7\sigma.$$

We shall prove (27) by the use of Lemma 2. Because of (20), there is a constant M_0 so that

$$|\hat{\mu}(n)| < 2\sigma \quad \text{or} \quad |\hat{\mu}(n) - 1| < 2\sigma \quad \text{for} \quad |n| > M_0.$$

We have also seen that

$$|\hat{\mu}_d(n)| \leq 5\sigma \quad \text{or} \quad |\hat{\mu}_d(n) - 1| \leq 5\sigma, \quad \text{all } n \in \mathbf{Z}.$$

Thus, since $\mu_c = \mu - \mu_d$,

$$(28) \quad |\hat{\mu}_c(n) + 1| < 7\sigma \quad \text{or} \quad |\hat{\mu}_c(n)| < 7\sigma \quad \text{or} \quad |\hat{\mu}_c(n) - 1| < 7\sigma \quad \text{for} \quad |n| > M_0.$$

Let λ be the continuous measure $\mu_c * \mu_c$, so $\hat{\lambda}(n) = \hat{\mu}_c(n)^2$, for all $n \in \mathbf{Z}$, and $\|\lambda\| \leq C^2$. Then

$$(29) \quad |\hat{\lambda}(n)| < 14\sigma \quad \text{or} \quad |\hat{\lambda}(n) - 1| < 14\sigma \quad \text{for} \quad |n| > M_0$$

follows from (28). By Lemma 2, the set $\{n: |\hat{\lambda}(n)| \geq 1/100\}$ is finite. But

$$\{n: |\hat{\lambda}(n)| \geq 1/100\} = \{n: |\hat{\mu}_c(n)| \geq 1/10\},$$

so that $\{n: |\hat{\mu}_c(n)| \geq 1/10\}$ is finite, and (27) is a consequence of (28). This completes the proof of Theorem 2.

3.3. We now proceed to the proof of Lemma 2. There is no loss of generality in assuming that the constant C is an integer. Also, we may assume λ to be a real measure, considering otherwise the measure $\eta = 2 \operatorname{Re} \lambda$, for which

$$(30) \quad \hat{\eta}(n) = \hat{\lambda}(n) + \overline{\hat{\lambda}(-n)}, \quad n \in \mathbf{Z}.$$

Thus, if we find $\gamma_1(C)$ for real measures, then

$$(31) \quad \gamma(C) = \frac{1}{2}\gamma_1(2C)$$

will work for arbitrary measures.

Because of Theorem 1, there is a constant $\sigma(C)$ so that if $\eta \in M(\mathbf{T})$ satisfies $\|\eta\| < C$ and

$$\limsup_{n \rightarrow -\infty} |\hat{\eta}(n)| \leq \sigma(C),$$

then

$$\limsup_{n \rightarrow +\infty} |\hat{\eta}(n)| < 1/10.$$

We define $\gamma = \gamma_1(C)$ by

$$(32) \quad \gamma_1(C) = \min(\sigma(C), Ce^{-10C}).$$

Suppose now that λ is a real continuous measure with $\|\lambda\| < C$ and satisfying

$$(33) \quad |\hat{\lambda}(n)| < \gamma \quad \text{or} \quad \operatorname{Re} \hat{\lambda}(n) > 1 - \gamma \quad \text{for} \quad |n| \geq N_0.$$

We shall prove that the set

$$(34) \quad \{n: |\hat{\lambda}(n)| \geq \gamma\}$$

is finite by assuming that it is infinite and deriving a contradiction. Since the set (34) is assumed infinite and (33) holds, the set

$$(35) \quad \{n: |n| > N_0, \operatorname{Re} \hat{\lambda}(n) > 1 - \gamma\}$$

is infinite. The measure λ is real, and thus

$$(36) \quad \hat{\lambda}(-n) = \overline{\hat{\lambda}(n)} \quad n \in \mathbf{Z}.$$

As a consequence, since (35) is infinite, the set Λ of positive integers defined by

$$(37) \quad \Lambda = \{n: n > N_0, \operatorname{Re} \hat{\lambda}(n) > 1 - \gamma\}$$

must also be infinite.

The measure λ is continuous, so the function $|\hat{\lambda}|^2$ has mean value 0 on \mathbf{Z} . (See p. 42 of [5].) As a consequence, the set Λ has density 0 in \mathbf{Z} . Thus it is possible to find a sequence $\{n_k: k = 1, 2, \dots\}$ of positive integers in Λ having the following property: For each k , the integers in the list

$$(38) \quad n_k - k, n_k - k + 1, n_k - k + 2, \dots, n_k - 1$$

are not in Λ . Because of (33) and (37), for every integer m occurring in one of the lists (38), $|\hat{\lambda}(m)| < \gamma$.

By taking a subsequence of the original sequence $\{n_k: k = 1, 2, \dots\}$ if necessary, we may assume that the sequences

$$\{e^{-in_k t} \lambda: k = 1, 2, \dots\}$$

in $M(T)$ converges in the weak* topology of $M(T)$ to a measure λ_0 . Since $e^{-in_k t} \lambda$ converges weak* to λ_0 in $M(T)$,

$$(39) \quad \lim_{k \rightarrow \infty} \hat{\lambda}(n + n_k) = \hat{\lambda}_0(n), \quad n \in \mathbf{Z}.$$

In other words, the sequence

$$\{\hat{\lambda}(\cdot + n_k): k = 1, 2, \dots\}$$

of translates of $\hat{\lambda}$ converges pointwise on \mathbf{Z} to the function $\hat{\lambda}_0$.

Because of (33) and (39), if m is any integer, one of the two alternatives

$$(40) \quad |\hat{\lambda}_0(m)| \leq \gamma$$

or

$$(41) \quad \operatorname{Re} \hat{\lambda}_0(m) \geq 1 - \gamma$$

must hold. We show next that the first alternative (40) must hold if m is either a negative integer or is a sufficiently large positive integer.

Suppose that m is negative. For each $k > |m|$, $n_k + m$ occurs in the list (38), and thus $|\hat{\lambda}(n_k + m)| < \gamma$. As a consequence, because of (39), $|\hat{\lambda}_0(m)| \leq \gamma$. This proves that

$$(42) \quad |\hat{\lambda}_0(n)| \leq \gamma, \quad n = -1, -2, -3, \dots$$

Assertion (42), together with (32), the definition of $\sigma(C)$ and the fact that $\|\lambda_0\| \leq \|\lambda\| < C$ shows that

$$\limsup_{n \rightarrow +\infty} |\hat{\lambda}(n)| < 1/10.$$

As a consequence, if m is a sufficiently large positive integer, the second alternative (41) is impossible, so (40) must hold. Because of our conclusions about $\hat{\lambda}_0(m)$ for both positive and negative m , we have the following: There is a positive integer N so that

$$(43) \quad |\hat{\lambda}_0(m)| \leq \gamma, \quad \text{all } m \in \mathbf{Z}, \quad |m| > N.$$

From (39) and (43) we conclude that, if m is an integer with $|m| > N$, then

$$|\hat{\lambda}(n_k + m)| < 2\gamma$$

if k is large enough. As a consequence, if $M > N$, for k sufficiently large,

$$|\hat{\lambda}(n_k + m)| < 2\gamma \quad \text{if } N < |m| \leq M.$$

Equivalently, if $M > N$, then

$$|\hat{\lambda}(n)| < 2\gamma \quad \text{if } N < |n - n_k| \leq M,$$

for k sufficiently large.

Using this fact, we can choose a finite sequence

$$\{m_j: j = 1, 2, \dots, K\}, \quad K = 100C^2,$$

from $\{n_k: k = 1, 2, \dots\}$ which has the following properties

$$(44) \quad m_1 > N;$$

$$(45) \quad m_j > 3m_{j-1};$$

$$(46) \quad |\hat{\lambda}(n)| < 2\gamma \quad \text{if} \quad N < |n - m_j| \leq \sum_{k=1}^{j-1} m_k.$$

We use the sequence $\{m_j: j = 1, \dots, K\}$ to construct the finite Riesz product

$$(47) \quad \varphi(t) = \prod_{j=1}^K \left(1 + \frac{i}{\sqrt{K}} \cos m_j t \right)$$

(see p. 107 of [5] for the basic properties of Riesz products.) Each term in the product (47) has sup norm no larger than $(1 + K^{-1})^{1/2}$, so

$$(48) \quad \|\varphi\|_\infty \leq (1 + K^{-1})^{K/2} < e.$$

Each term in the product (47) has norm $1 + K^{-1/2}$ in the Banach space $A(T)$ of functions having absolutely convergent Fourier series. Thus

$$(49) \quad \|\varphi\|_{A(T)} \leq (1 + K^{-1/2})^K \leq e^{\sqrt{K}} = e^{10C}.$$

We shall obtain our contradiction by integrating the Riesz product φ with respect to the measure λ . Because of (48) and $\|\lambda\| \leq C$,

$$(50) \quad \left| \int_T \varphi d\lambda \right| \leq \|\varphi\|_\infty \|\lambda\| \leq eC.$$

The Parseval formula gives

$$(51) \quad \begin{aligned} \left| \int_T \varphi d\lambda \right| &= \left| \sum_{n \in \mathbf{Z}} \hat{\varphi}(n) \hat{\lambda}(-n) \right| \\ &\geq \left| \sum_{j=1}^K (\hat{\varphi}(m_j) \hat{\lambda}(-m_j) + \hat{\varphi}(-m_j) \hat{\lambda}(m_j)) \right| - \left| \sum_{n \in J} \hat{\varphi}(n) \hat{\lambda}(-n) \right|, \end{aligned}$$

where J is the complement of $\{m_1, -m_1, \dots, m_K, -m_K\}$ in \mathbf{Z} . As a consequence of condition (45), there will be no cancellat on when the terms of the product (47) are multiplied out, so

$$(52) \quad \hat{\varphi}(m_j) = \hat{\varphi}(-m_j) = \frac{i}{2} K^{-1/2}.$$

Because each m_j is in Λ , which has been defined by (37), and (36) holds,

$$(53) \quad \operatorname{Re} \hat{\lambda}(m_j) = \operatorname{Re} \hat{\lambda}(-m_j) > 1 - \gamma, \quad j = 1, \dots, K.$$

Using (52) and (53), we see that

$$(54) \quad \begin{aligned} &\left| \sum_{j=1}^K (\hat{\varphi}(m_j) \hat{\lambda}(-m_j) + \hat{\varphi}(-m_j) \hat{\lambda}(m_j)) \right| \\ &\geq \sqrt{K}(1 - 2\gamma) = 10C(1 - 2\gamma). \end{aligned}$$

If n is an integer in J with $\hat{\phi}(n) \neq 0$, then, because of (47), n must be of the form

$$\sum_{j=1} \varepsilon_j m_j, \quad \varepsilon_j = 0, 1 \text{ or } -1,$$

with $\varepsilon_j \neq 0$ for at least two values of j . Because of (44), (45), (46) and (36), for such an integer n ,

$$|\hat{\lambda}(-n)| < 2\gamma.$$

Thus, by (49),

$$\begin{aligned} (55) \quad \left| \sum_{n \in J} \hat{\phi}(n) \hat{\lambda}(-n) \right| &\leq 2\gamma \sum_{n \in J} |\hat{\phi}(n)| \\ &\leq 2\gamma \|\hat{\phi}\|_{A(T)} \leq 2\gamma e^{10C}. \end{aligned}$$

Combining (50), (51), (54) and (55), we obtain

$$(56) \quad eC \geq 10C(1 - 2\gamma) - 2\gamma e^{10C} = 10C - 20C\gamma - 2\gamma e^{10C}.$$

Because of (32), $\gamma \leq Ce^{-10C}$, and thus (56) yields

$$eC \geq 10C - 20C^2 e^{-10C} - 2C,$$

which gives the contradiction

$$e \geq 8 - 20Ce^{-10C}.$$

This completes the proof of Lemma 2.

3.4. We indicate here two ways in which Theorem 2 can be extended.

First, by applying Theorem 1 to the measure $\mu - \mu * \mu$, the assumption

$$\limsup_{|n| \rightarrow \infty} |\hat{\mu}(n) - \hat{\mu}(n)^2| < \delta$$

can be replaced by the one-sided assumption

$$\limsup_{n \rightarrow +\infty} |\hat{\mu}(n) - \hat{\mu}(n)^2| < \delta',$$

for appropriate δ' .

Second, Theorem 2 has as consequence a proposition bearing the same relation to it as Corollary 1 does to Theorem 1. The proposition states that, subject to a bound on $\|\mu\|$, if

$$\limsup_{|n| \rightarrow \infty} |(\hat{\mu}(n) - \alpha_1) \cdots (\hat{\mu}(n) - \alpha_m)|$$

is small enough, then $\hat{\mu}$ is close to a periodic function on \mathbf{Z} taking only the values $\{\alpha_1, \dots, \alpha_m\}$. The proposition can be proved by reduction to Theorem 2 in a

manner analogous to the way in which Corollary 1 was reduced to Theorem 1. We omit the details.

4. EXAMPLES

The aim of this section is the construction of the two examples of measures mentioned in the introduction. The first, μ_2 , satisfies

$$(57) \quad \limsup_{n \rightarrow +\infty} |\hat{\mu}_2(n)| \neq \limsup_{n \rightarrow -\infty} |\hat{\mu}_2(n)|.$$

The second, μ_4 , shows that the constant δ of Theorem 2 cannot be chosen to be independent of the bound C .

Let $n_j = 4^j + 1$ and μ_1 be the measure corresponding to the Riesz product

$$\prod_{j=1}^{\infty} (1 - \sin n_j t).$$

(See p. 167 of [5] for a discussion of Riesz products.) Then

$$\begin{aligned} \hat{\mu}_1(0) &= 1, & \hat{\mu}_1(n_j) &= i/2, & \hat{\mu}_1(-n_j) &= -i/2, \\ |\hat{\mu}_1(n)| &\leq 1/4, \text{ all other } n. \end{aligned}$$

Let $\mu_2 = \delta_0 - 2i\mu_1$, where δ_0 is the unit point mass at 0. Then

$$\begin{aligned} \hat{\mu}_2(0) &= 1 - 2i, & \hat{\mu}_2(n_j) &= 2, & \hat{\mu}_2(-n_j) &= 0, \\ |\hat{\mu}_2(n)| &\leq 3/2, \text{ all other } n, \end{aligned}$$

which proves (57).

Taking η to be normalized Lebesgue measure on T , so

$$\hat{\eta}(0) = 1, \hat{\eta}(n) = 0 \text{ if } n \neq 0,$$

we define μ_3 by

$$\mu_3 = 1/2(\mu_2 - (1 - 2i)\eta).$$

Then

$$\hat{\mu}_3(0) = 0, \hat{\mu}_3(n_j) = 1, |\hat{\mu}_3(n)| \leq 3/4, \text{ all other } n.$$

Let $\varepsilon > 0$. Choose m so $(3/4)^m < \varepsilon$. If μ_4 is the m -fold convolution of μ_3 with itself, then $\hat{\mu}_4 = (\hat{\mu}_3)^m$, so

$$\begin{aligned} \hat{\mu}_4(n_j) &= 1, \quad j = 1, 2, \dots, \\ |\hat{\mu}_4(n)| &< \varepsilon, \text{ all other } n. \end{aligned}$$

Since $n_j = 4^j + 1$, this shows that the constant δ of Theorem 2 cannot be chosen to be independent of the bound C .

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THE HEBREW UNIVERSITY OF JERUSALEM